Offsets of Cassini ovals

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Abstract

We report on computing and visualizing the offsets of the Cassini ovals. We use the elimination technique to obtain offsets as envelopes of a family of planar curves in a recent experimental version of GeoGebra. Offset curves are obtained in their implicit form as irreducible polynomials of degree 12 and 16.

1 Introduction

The study of plane algebraic curves is a domain which has been explored for centuries. Numerous results on the topic, description of curves form different points of view, are widely available, either in books such as [4] or in websites, for example `mathcurve.com` or `https://mathshistory.st-andrews.ac.uk/Curves/`. Up to the present, new features can still be discovered and explored. Technology is of great help for this and the newly developed, and still under development, tools for Automated Reasoning [16, 19] are central for the exploration and discovery of new theorems. They are based at a great extent on the theory of Gröbner bases and its various developments, on elimination ideals and resultants; see [5].

Many results have been conjectured using a Dynamic Geometry System (DGS), and proofs have been based on work assisted by a Computer Algebra System (CAS). It happens that a DGS includes a CAS; this is the case with GeoGebra, in which Giac is embedded [17]. Nevertheless, the discussion between DGS and CAS is still a dialog between different technologies [13, 20]. Specific developments have appeared where the dynamics of a DGS provide a strong visualization of the geometric situation, when the CAS provides a set of static snapshots, together with exact equations.

Another contribution of the CAS is via its package for polynomial rings. Sometimes, the automated commands of a DGS for the determination of geometric loci and envelopes provide a curve and a polynomial equation. The straightforward way to have a proof of the reducibility or irreducibility of the obtained curve is only by symbolic factorization. Hence the utility of having a CAS at hand.

A well known family of planar algebraic curves is defined by the so called Cassini ovals, which can be described in several ways, as we do so in what follows. However, the present work is devoted to the exploration of offsets of Cassini ovals by using several technological tools that surely the reader is aware of. Our explorative investigation provides a kind of exhaustive description of the offsets under study.

2 Cassini ovals

A *Cassini oval* is a plane curve $\mathcal{C}$ defined as follows.

Definition 1 *Take two distinct points F_1 and F_2 in the plane and a positive real b. Denote $a = F_1F_2$. The geometric locus of points M in the plane such that $MF_1 \cdot MF_2 = b^2$, if it is not empty, is called a Cassini oval.*

The points F_1 and F_2 are called the foci *of $\mathcal{C}$ and the number $e = b/a$ is called the* eccentricity *of the Cassini oval $\mathcal{C}$..*

Easy algebraic manipulations yield an implicit equation for $\mathcal{C}$.

Take $F_1(a, 0)$ and $F_2(-a, 0)$ for a given $a > 0$, and a generic point $M(x, y)$. Then the condition given in Def. 1 writes:

$$\sqrt{(x-a)^2 + y^2} \cdot \sqrt{(x+a)^2 + y^2} = b^2.$$

Squaring both sides, we obtain:

$$a^4 - 2a^2x^2 + 2a^2y^2 + x^4 + 2x^2y^2 + y^4 = b^4,$$

and this equation can be written as follows:

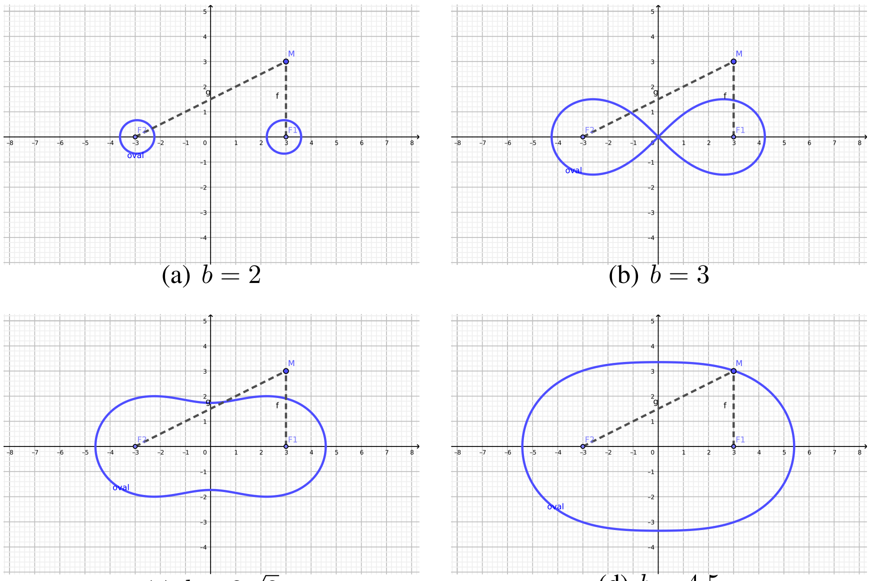
$$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) + a^4 - b^4 = 0, \tag{1}$$

showing that a Cassini oval is a *quartic*.

Figure 1 shows examples, for fixed foci ($a = 3$) and various values of b.

The similarity with ellipses is not a coincidence. Giovanni Domenico Cassini (1625-1712) was an Italian born and French naturalized mathematician, engineer and astronomer. He discovered four satellites of the giant planet Saturn and thought that these ovals provided an accurate model for their orbits. A European probe called Cassini has been launched in 1997; it was the fourth probe to visit Saturn and the first to orbit this planet for years. It launched the lander Huygens to Titan, one of the most remarkable satellites of Saturn. Figure 2 shows a photo shot by the Cassini probe; the main gap within Saturn's rings system is also named after Cassini.

After Kepler's model for planetary motion was finally accepted, late in the 17$^{\text{th}}$ century, it is well known that planets describe elliptic orbits around the Sun, with the Sun at one of the foci. The same model is valid for satellites (either natural or artificial) orbiting a planet. In general, the problem of finding the trajectory of an object under the action of a central force (such as gravitational attraction), and obeying Newton's Second Law of movement, is known as the Kepler's problem. The mathematical model for this problem is provided by a second order differential equation whose solutions describe the trajectory of the object. It turns out that these solutions are conic sections, and if they are closed, then the orbit of the object is an ellipse with one of the foci located at the punctual mass, which bears the central force.



(a) $b = 2$ (b) $b = 3$

(c) $b = 2\sqrt{3}$ (d) $b = 4.5$

Figure 1: Examples of Cassini ovals

It has been claimed that Cassini ovals remained pure geometrical objects, but this is not fully accurate. Cassini ovals appear, for example, in electrostatics. Two plane curves are said to be *orthogonal* if they have orthogonal tangents at each point of intersection. Thus, the families of hyperbolas defined by the polynomials $u(x, y) = x^2 - y^2 - c$ and $v(x, y) = xy - c$, where $c \in \mathbb{R}$, are families of orthogonal curves (see Figure 3), which is easily proven with elementary Calculus. The foci being given, the orthogonal trajectories of the Cassini ovals with these foci are rectangular hyperbolas, describing the field lines of an electrostatic field. See `https://mathcurve.com/courbes2d.gb/cassini/cassini.shtml` for a short description.

Cassini ovals appear also in optics as *isoptic curves* of conics, sometimes under their other name *Spiric curves* [7, 8, 12]. We recall that given a plane curve C and an angle θ, the θ-isoptic of C (if it exists) is the geometric locus of points through which passes a pair of tangents forming an angle equal to θ. For conics, in fact for any smooth strictly convex curve, such isoptics exist for any θ. For other curves, especially non convex curves, isoptics may exist for specific angles only. Examples where not every angle provides an isoptic are given in [9, 10].

A very important geometric feature of Cassini ovals is that they can be obtained as *toric intersections*, namely the intersection of a torus with a plane parallel to the torus axis. Figure 4 shows various examples, according to the distance d from the axis to intersecting plane. Note that in the second example, (Figure 4b) we obtain a Bernoulli lemniscate. In all these examples, the radius of the revolving circle is $r = 2$, the distance from the center of the generating circle to the axis is $R = 4$.

The reverse engineering to determine the pair torus-plane from the equation of the Spiric curve is described in detail in [7] and a companion dynamic applet[1] is available at `https://www.geogebra.org/m/xBeqhrtD`. If the torus is self-intersecting, the plane-torus intersection may have either

[1] The applet has been developed with GeoGebra, a free Dynamic Geometry System at `geogebra.org`

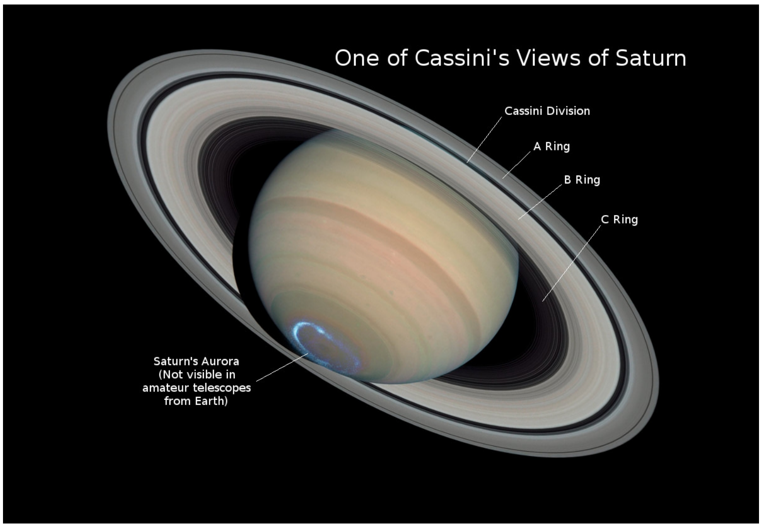


Figure 2: Saturn and its rings with Cassini Division. Credit: NASA/JPL/Space Science Institute

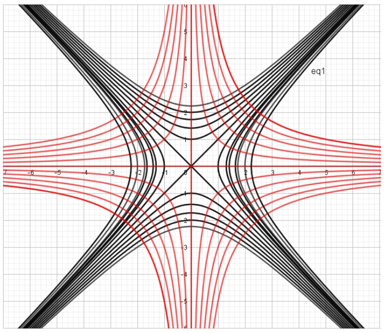


Figure 3: Two orthogonal families of curves

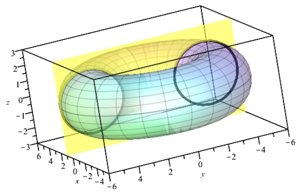
one component, or two disjoint components, as shown in Figure 5. Here the example corresponds to $R = 5, r = 3$. This is exactly the case for bisoptic curves of conics. Even if the bisoptic curves have two disjoint components, the defining polynomial of degree 4 is irreducible.

Before closing this section, we remark that it is often useful to use the parametrical presentation of a Cassini oval given in [14], namely:

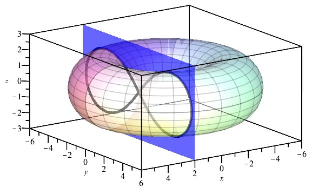
$$\begin{cases} x(t) & = \frac{b^4-t^4}{4at^2}, \\ y(t) & = \pm\sqrt{t^2-(x-a)^2} = \pm t\sqrt{1-\left(\frac{b^2-4a^2t^2-t^4}{4at^3}\right)^2}. \end{cases} \tag{2}$$

An interactive GeoGebra applet is available at https://www.geogebra.org/m/frpu4c4z for the construction of Cassini ovals using Eq. (2). Changing the parameters a and b with sliders enable to visualize the different shapes. The same applet will be useful in the next section to visualize the offsets.

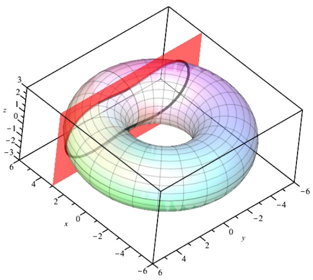
As an additional example, we mention that Cassini ovals can be easily created by using four-bar linkages. If we create a linkage as shown in Figure 6 (also available as a web applet at https://www.geogebra.org/m/qe699ja6) we can fix points A and B, and by fine-tuning the



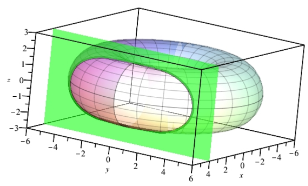
(a) $d = 1$



(b) $d = 2$

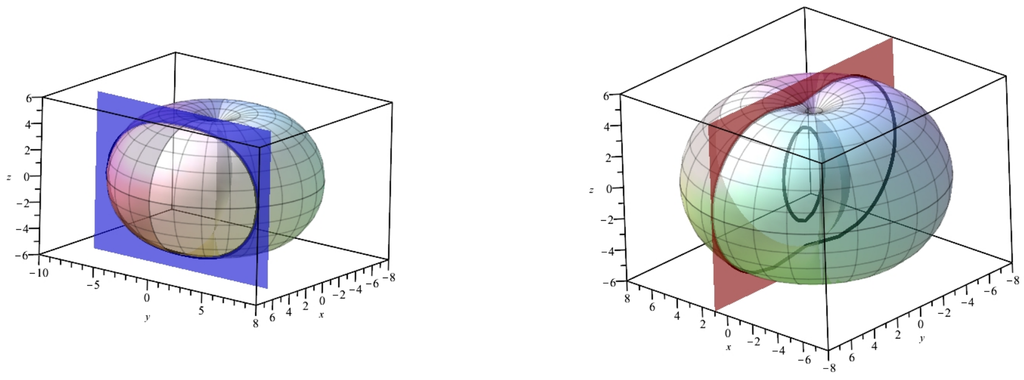


(c) $d = 3$



(d) $d = 4$

Figure 4: Cassini ovals as toric intersections



(a) $d = 4$ (b) $d = 1$

Figure 5: One or two components in toric intersections

lengths BC, CD and AD we can check how the midpoint E of CD changes while the quadrilateral $ABCD$ is moving.

This motion is inspired by a funny fictional story of a bug that sits on the top of a toothpaste box and swings to left and right. We work with a 2D model of the motion (see the top illustration of Figure 6) which depicts a parallelogram. Now let us put $A = (0, 0)$, $B = (4, 0)$, and set $BC = AD = 3$ and $CD = 4$. If $E = (x, y)$ describes the position of the bug (which is the midpoint of CD), then, by eliminating all coordinates but x and y, one gets the equation $x^6 - 12x^5 + 3x^4y^2 + 42x^4 - 24x^3y^2 - 16x^3 + 3x^2y^4 + 52x^2y^2 - 111x^2 - 12xy^4 - 16xy^2 + 60x + y^6 + 10y^4 - 95y^2 + 100 = 0$. In fact, this factorizes to a circle and a quartic, and the latter one is a Cassini oval.

We conclude algebraically that the bug's motion is not just a circular movement but also a Cassini oval. Geometrically, however, this seems to be confusing: only the circular movement should be accepted. It is clear why a circular motion appears: both C and D follow a circular motion with the same radius and phase, so their midpoint will follow another circular motion with the same radius and phase, too. On the other hand, we cannot describe algebraically the precondition that the quadrilateral is convex, so, in fact the situation that $ABCD$ describes a parallelogram, cannot be ensured. Actually, the case that $ABCD$ is an antiparallelogram (see the applet screenshot in Figure 6) is also contained in the algebraic description. That is why we obtain an extra component. When trying out the applet, we learn that there is a degenerate position of the parallelogram (when $C = (7, 0)$ and $D = (3, 0)$) which is, at the same time, a degenerate position of the antiparallelogram too—so the bug is able to switch from a circular motion to a quartic one, if the physical movement of the linkage allows crossed bars.

Technically speaking, using Gröbner bases and elimination we obtain a sextic curve that consists of the circular component and the quartic as well. (See also [23, p. 38–42] for more details.)

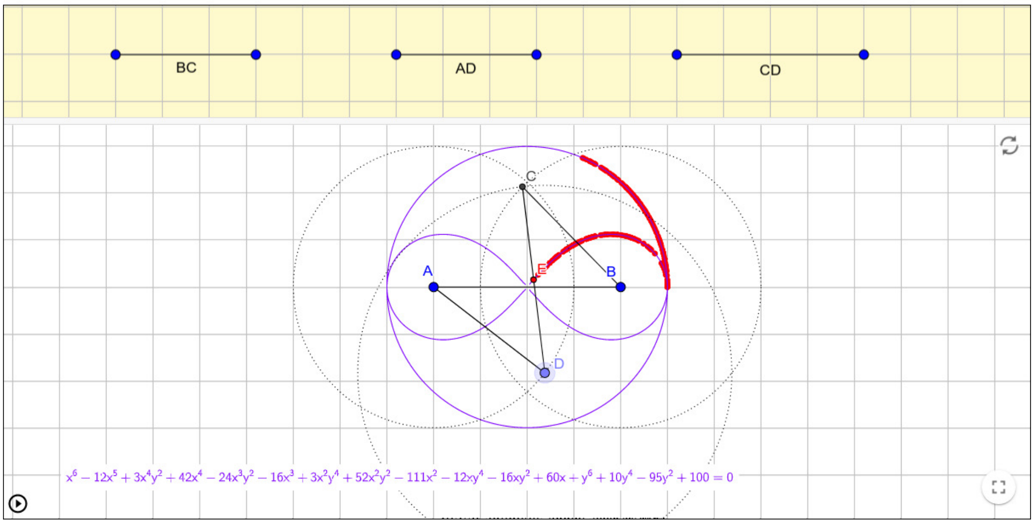


Figure 6: One midpoint of a moving antiparallelogram traces a Cassini oval

3 Envelopes and offsets

3.1 Envelope of a 1-parameter family of plane curves

There exist several definitions of envelopes in the literature. For example in [4], four non-equivalent definitions are presented . Three of them are listed by Kock in [15] and he calls these synthetic, impredicative and analytic, respectively. Here we use his *analytic definition*, which is the only one given by Berger in [2].

Definition 2 *Denote by $\mathcal{C}_k$ a plane curve given by an equation of the form $F(x, y, k) = 0$, where k is a real parameter. An envelope of the family $\{\mathcal{C}_k\}$, if it exists, is the geometric locus of points verifying the following system of equations:*

$$\begin{cases} F(x, y, k) = 0, \\ \frac{\partial F}{\partial k} F(x, y, k) = 0. \end{cases}$$

We will denote by $\mathcal{E}$ an envelope according to this definition.

Example 3 *It is well known that the parabola $y = x^2$ has a tangent line at each point (k, k^2), and the corresponding tangent line can be described with the formula $y = (2k)x - k^2$. Let us consider the set of tangents $\mathcal{C}_k$ of the form $F(x, y, k) = y - ((2k)x - k^2) = 0$. Now, by computing $\frac{\partial F}{\partial k} F(x, y, k) = -2x + 2k$ we need to solve the equation system of Definition 2. It yields $x = k$ and $y = x^2$, and the latter equality confirms that the envelope of the family $\mathcal{C}_k$ is indeed the parabola $y = x^2$. (See Figure 7 and `https://www.geogebra.org/m/m4pydrdd` for an interactive applet.)*

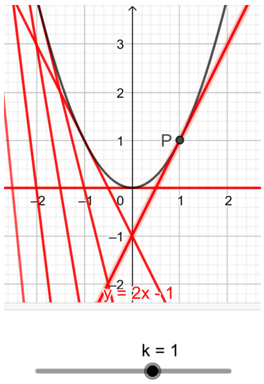


Figure 7: The envelope of the family of lines $y = (2k)x - k^2$ is the parabola $y = x^2$.

3.2 Offsets

Definition 4 *Let $\mathcal{C}$ be an irreducible real algebraic plane curve, and let $\mathcal{C}_0 \subset \mathcal{C}$ be the set of regular points P of $\mathcal{C}$ i.e. points where the norm, in the metric affine space $\mathbb{R}^2$, of the normal vector is non-zero. Then, the* offset *to $\mathcal{C}$, at distance d, is the Zariski closure of the intersection points of the circles of radius d centered at each point $P \in \mathcal{C}_0$ and the normal line to $\mathcal{C}$ at p.*

We will denote by $\mathcal{O}_d$ the offset according to this definition. Bruce and Giblin [4] show that $\mathcal{O}_d \subset \mathcal{E}$. A more formal definition can be found in [21].

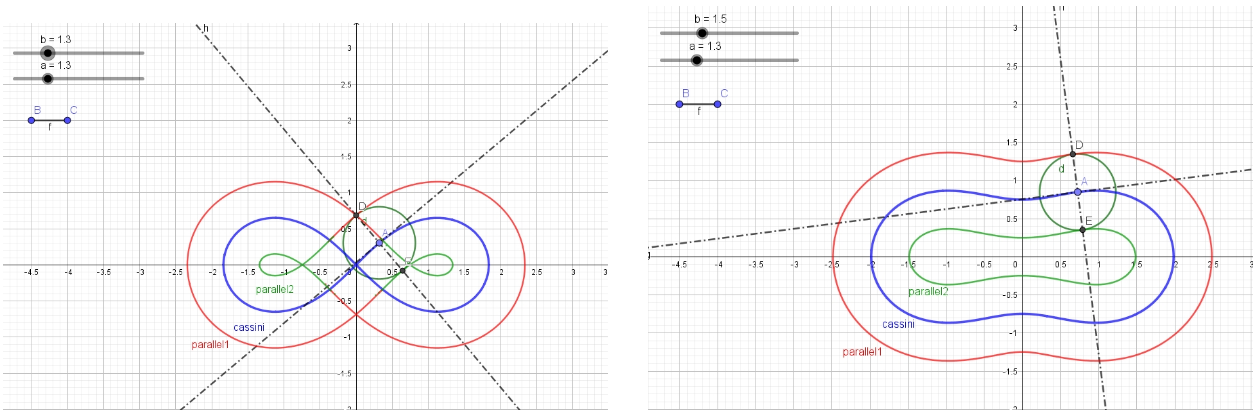
Example 5 *Let $\mathcal{C}$ be the x-axis. Then, the offset to $\mathcal{C}$, at distance 1 is the union of lines $y = 1$ and $y = -1$, that is, the geometric set of the algebraic equation $y^2 - 1 = 0$.*

Offsets of plane curves have been studied for a quite long time. Formulas for the degree of offset of plane curves have been derived in [21, 22]. Offsets of various curves have been studied in the past by the authors of the present paper: offsets of conics in [7], offsets of an astroid in [6, 11], offset of a deltoid in [13]. As mentioned, an offset is a particular case of an envelope of a 1-parameter family of plane curves.

4 Offsets of Cassini ovals

4.1 Dynamical exploration with a DGS

In this subsection, the notations will be those of Figure 8. The segment BC determines the distance f which will be the offset distance. This construction is enabled by performing geometric constructions, by applying the **Locus** command. We consider a Cassini oval (the blue curve). At each point A on the curve, the normal at A has two points D and F at the given distance f. The geometric locus of these points is called a parallel curve to the Cassini oval, or the offset at distance f. Letting f vary, we obtain different shapes for the Cassini oval, either convex or not, either self intersecting or not. Figure 8 shows two possible shapes and also the corresponding offsets according to the original Cassini oval. For example, in Figure 8(a), the oval is actually the lemniscate of Bernoulli. As mentioned above, in each case, the offset has been obtained using the **Locus** command of GeoGebra; it provided a graphical output, but no algebraic equation. Note the shape of the offset: it is not the union of two disjoint components, but actually a single continuous curve (note the curves parallel1 and parallel2 in the figures).



(a) (b)

Figure 8: Parallels to Cassini Ovals

4.2 Algebraic work

We consider the family of Cassini ovals given by the implicit quartic equation (1), where a and b are positive real parameters. Using the DGS GeoGebra, the presence of the two parameters a and b requires the definition of two slider bars. Here we emphasize that a very recent experimental version of GeoGebra (namely, *GeoGebra Discovery*, available at `https://github.com/kovzol/geogebra-discovery`) is required to visualize all steps below.

Figure 9 shows the offset curve for the case $a = b = d = 1$. The curve is obtained via elimination, that is, after describing the algebraic position of the circle with radius d, and considering an envelope curve that is tangent to all such circles c. In fact, we use Definition 2 to obtain the envelope algebraically, by using Eq. (1) as the implicit definition of the Cassini oval. The corresponding curve is denoted by $eq1$ in Figure 9. Now we get the equation of the curve denoted by $eq2$ in Figure 9, as follows:

$$
\begin{aligned}
&(x^2+y^2-1)\cdot(x^{12}+6x^{10}y^2-12x^{10}+15x^8y^4-48x^8y^2+52x^8+20x^6y^6-72x^6y^4\\
&+112x^6y^2-114x^6+15x^4y^8-48x^4y^6+72x^4y^4-118x^4y^2+132x^4+6x^2y^{10}-12x^2y^8\\
&+16x^2y^6-86x^2y^4+64x^2y^2-72x^2+y^{12}+4y^8-18y^6-4y^4+20y^2+9)=0,
\end{aligned}
$$

which is already a factorized form of the obtained algebraic equation.

The first factor describes the unit circle, shown with a thin linestyle in Figure 9. It can be disputed whether this circle belongs to the offset geometrically or not - but even if we are disappointed by getting this degenerate component, it is automatically obtained by elimination; see [18]. The other factor is more useful: it is of degree 12 and describes exactly what we expect.

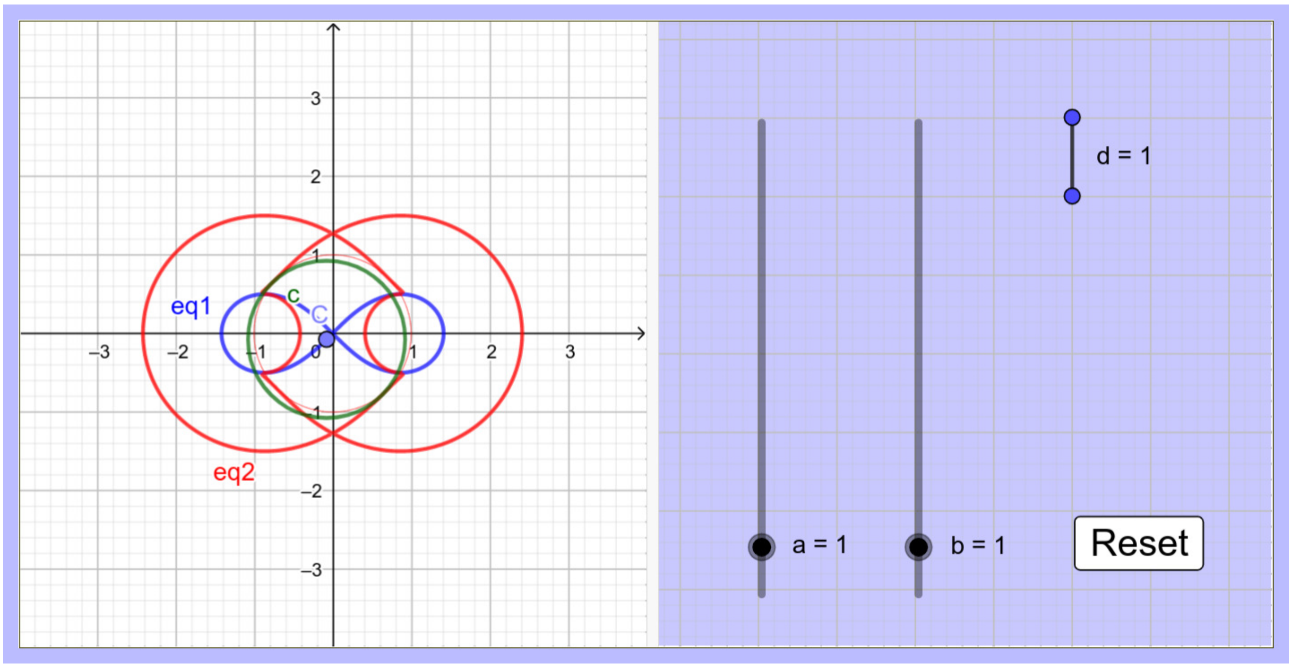


Figure 9: Offset of a Cassini oval with parameters $a = b = d = 1$

In this section we discuss all possible cases of the triplets $(a, b, d) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$, in an experimental way. To do that, we simply move the slider bars and try to identify all cases that differ algebraically or geometrically. Without loss of generality, we can assume that a or b may be fixed, because a similarity transformation can be used to get the same geometrical setup. On the other hand, the degenerate cases $a = 0$, $b = 0$ or $d = 0$ can still be investigated by moving the slider. So we discuss these limiting cases first.

1. If $a = b = 0$, we get $(x^2 + y^2)^2 = 0$ which is geometrically the point $(0, 0)$ for the degenerate Cassini curve. Here the elimination process leads to the equation $0 = 0$ which describes the whole plane, but this does not make much sense geometrically.

2. If $a = 0$, $b \neq 0$, Eq. (1) simplifies to $x^2 + y^2 = b^2$, an equation of a circle with radius b. Depending on d, different offset curves can be obtained:

   (a) If $d \neq b$, a set of two circles (centered in the origin, with radii $|b \pm d|$) and an empty quartic component appear.

   (b) If $d = b$, theoretically one circle (centered in the origin) should appear, but, elimination returns an empty set—this seems to be an implementation issue in GeoGebra Discovery.

3. If $a \neq 0$, $b = 0$, then Eq. (1) simplifies to $(x^2 + y^2)^2 - 2a^2(x^2 - y^2) + a^4 = 0$. The left-hand side is strictly greater than $(x^2 + y^2)^2 - 2a^2(x^2 + y^2) + a^4 = ((x^2 + y^2) - a^2)^2$ which is non negative. Therefore on this case, the Cassini curve is empty.

4. In all remaining cases, we can assume that $a = 1$.

(a) We already observed the case $b = 1$, for $d = 1$. For different values of d such that $d \neq 0$, the output has the same algebraic structure as for $d = 1$: two factors appear, a quadratic one and one of degree 12. The quadratic factor is $x^2+y^2-d^2$, which corresponds to a circle (centered at the origin, with radius d). We note that this case describes the lemniscate of Bernoulli.

(b) If $b \neq 1$, all other types of Cassini ovals will be observed. In all cases we get a polynomial of degree 16, and it is not reducible over $\mathbb{Z}[x, y]$. We learn this by trusting GeoGebra's internal computer algebra system, Giac.

Giac is a freely available CAS. Its name echoes the abbreviation "Giac is a computer algebra system", defined recursively infinitely. Giac has been developed since 2000 and maintained by Bernard Parisse from the University of Grenoble I, France, and is well-known as the embedded computer algebra system in HP calculators. It is written in C++ and provides programming interfaces for PHP, Java, JavaScript and Python. It has fast implementation of algorithms for polynomial operations like multiplication, division, the greatest common divisor of multivariate polynomials, and Gröbner basis computations.

Giac is actually the main player when computing the elimination in a very efficient way: so it is possible to drag the sliders and get an immediate feedback on the screen. However, it is not possible in GeoGebra leads to the polynomial equation

$$
\begin{aligned}
&x^{16} + 8x^{14}y^2 - 48x^{14} + 28x^{12}y^4 - 384x^{12}y^2 + 1000x^{12} + 56x^{10}y^6 - 1296x^{10}y^4 \\
&+ 9264x^{10}y^2 - 16224x^{10} + 70x^8y^8 - 2400x^8y^6 + 30648x^8y^4 - 161536x^8y^2 + 161296x^8 \\
&+ 56x^6y^{10} - 2640x^6y^8 + 49952x^6y^6 - 455488x^6y^4 + 1715520x^6y^2 - 1411200x^6 \\
&+ 28x^4y^{12} - 1728x^4y^{10} + 43608x^4y^8 - 562304x^4y^6 + 3794656x^4y^4 - 12548352x^4y^2 \\
&+ 6762240x^4 + 8x^2y^{14} - 624x^2y^{12} + 19632x^2y^{10} - 323168x^2y^8 + 3146304x^2y^6 \\
&- 18370176x^2y^4 + 42126336x^2y^2 - 31352832x^2 + y^{16} - 96y^{14} + 3592y^{12} - 71040y^{10} \\
&+ 902032y^8 - 7418880y^6 + 24215040y^4 - 33177600y^2 + 16257024 = 0.
\end{aligned}
$$

We utilized Maple's `evala(AFactor(...)))` command to double check irreducibility. It seems very likely that all cases lead to an irreducible polynomial (of degree 16) over the reals.

We note that the obtained polynomial curves, all of degree 16, show a very rich geometrical variety. To illustrate this diversity we show some of the offsets in Figure 11. All of these outputs were generated by using the GeoGebra applet at https://matek.hu/zoltan/offsets-of-Cassini-ovals.php.

5 Conclusion

We have summarized several well-known properties of Cassini ovals, including some of their possible formalizations. Also we have computed the offset curves of some Cassini ovals for various parameters. By using several applets made with the aid of DGSs, we have provided visual evidence that supports several claims we have made throughout the paper. We also have learnt, based on that evidence, that the offset curves we investigated are of degree 12 or 16. However, we did not provide a completely rigorous proof for all the cases.

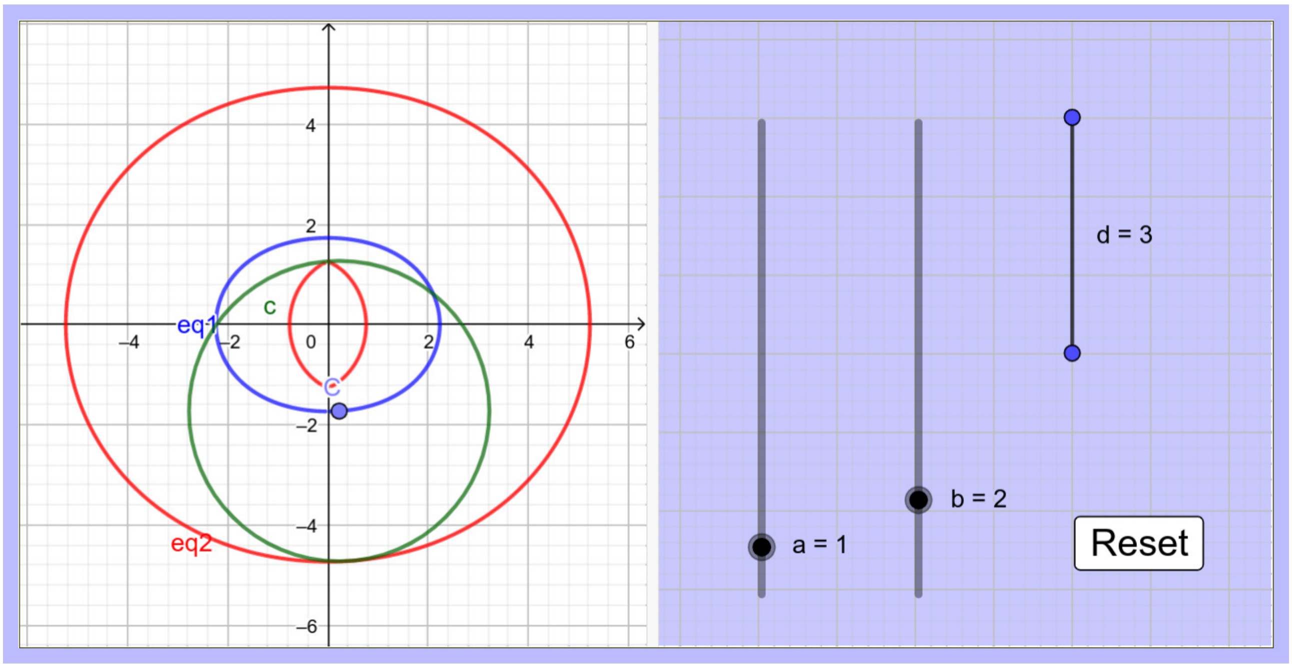


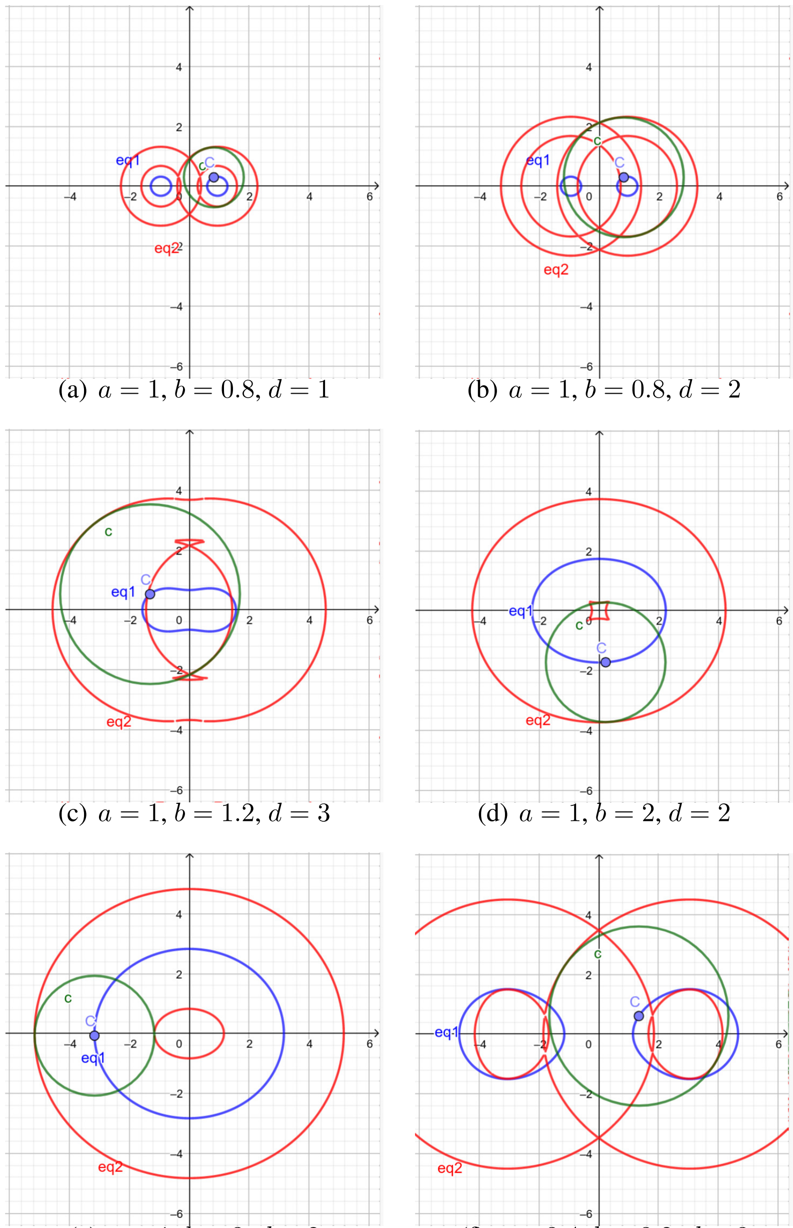
Figure 10: Offset of a Cassini oval with parameters $a = 1$, $b = 2$, $d = 3$

We emphasize that the method we used is heavily based on recent computer algebra techniques, including elimination via Gröbner bases, and their fast implementation in the GeoGebra/Giac computation network. The reader is motivated to try her or his own experiments and obtain a large set of—to our knowledge—previously unknown curves.

As shown by the technology applets we have presented throughout this work, we have made an extensive use of the parametrical representations for plane curves. It is well known that parametric plot is generally more accurate than implicit plot. This issue has been discussed in [25] for functions of two real variables; the problems are similar for plane curves, in a less acute way.

Moreover, a parametric presentation of a given plane curve is not unique. When both a trigonometric presentation and a rational presentation exist, their efficiencies are different, according to the purpose of the computations. If an animation is built, then in most cases, a trigonometric parametrization yields smoother displacements along the curve; see, for example, [6, 11]. In the present work, we were more interested in plotting the curve and not in animations. Therefore we insisted more on the implicit equations of the curves and on factorization. The dynamical aspects can be experienced using the online applets whose URL are given in the text.

Finally we wish to emphasize that information is well known for plane curves of low degree. For example, a complete catalog exists for degrees 2 and 4. Numerous questions are still open for higher degrees. The kind of explorative studies, having a strong experimental aspect that we have described here, and are found in other papers [10, 11], enable one to have a deep insight into curves of much higher degrees.



(a) $a = 1, b = 0.8, d = 1$ (b) $a = 1, b = 0.8, d = 2$

(c) $a = 1, b = 1.2, d = 3$ (d) $a = 1, b = 2, d = 2$

(e) $a = 1, b = 3, d = 2$ (f) $a = 3.4, b = 3.2, d = 3$

Figure 11: Geometrical diversity of offsets of Cassini ovals with different parameters

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The "footpaste" illustration in Figure 6 was rendered by Benedek Kovács by using the Blender 3D computer graphics software toolset.

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